

# Rationality of some Gromov-Witten varieties and application to quantum $K$ -theory

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## Abstract

We show that for any minuscule or cominuscule homogeneous space  $X$ , the Gromov-Witten varieties of degree  $d$  curves passing through three general points of  $X$  are rational or empty for any  $d$ . Applying techniques of A. Buch and L. Mihalcea [BuMi08] to constructions of the authors together with L. Manivel [ChMaPe08], we deduce that the equivariant  $K$ -theoretic three points Gromov-Witten invariants are equal to classical equivariant  $K$ -theoretic invariants on auxilliary spaces again homogeneous under  $\text{Aut}(X)$ .

## Introduction

To compute Gromov-Witten invariants in cominuscule Grassmannians of classical type, A. Buch, A. Kresch, and H. Tamvakis proved a comparison formula which they called “quantum to classical principle” [Buc03, BuKrTa03]. This formula compares Gromov-Witten invariants on a homogeneous space  $G/P$  with classical intersection numbers on an auxilliary homogeneous space under the same group  $G$ . In [ChMaPe08], L. Manivel and us gave a uniform treatment of their results in order to include the two exceptional homogeneous spaces  $E_6/P_1$  and  $E_7/P_7$ . This “quantum to classical principle” was further generalised by A. Buch and L. Mihalcea to the  $K$ -theoretic invariants in [BuMi08], in the case of Grassmannians. In this paper we show such a principle for all cominuscule homogeneous spaces: see Corollary 3.6.

As it is explained in [BuMi08], such a result is a consequence of the fact that the locus  $GW_d(x_1, x_2, x_3)$  of curves of degree  $d$  passing through three generic points  $x_1, x_2, x_3$  is rational. Denoting  $d_{\max}$  the minimal integer  $d$  such that there passes a curve of degree  $d$  through any two points in  $X$ , they made the following conjecture:

**Conjecture 0.1 (Buch-Mihalcea)** Let  $X$  be a cominuscule variety and let  $d > d_{\max}$ . If  $x_1, x_2, x_3$  are general points in  $X$ , then the Gromov-Witten variety  $GW_d(x_1, x_2, x_3)$  is rational.

This conjecture is true in type  $A$  (already if  $d \geq d_{\max}$ ) by [BuMi08, Corollary 2.2], and we show that it is true for any cominuscule space except exactly in the case where  $X = E_6/P_1$  and  $d = 3$ . To cover this case we introduce the following notation: let  $X_d^{(3)} \subset X^3$  denote the subvariety of triples of points  $(x_1, x_2, x_3)$  such that there exists a stable curve of degree  $d$  through  $x_1, x_2, x_3$ . Proposition 3.4 contains in particular:

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**Theorem 0.2** *Let  $X$  be a cominuscule variety, and  $d$  be any integer. Then for  $(x_1, x_2, x_3)$  generic in  $X_d^{(3)}$ , the Gromov-Witten variety  $GW_d(x_1, x_2, x_3)$  is rational.*

As a consequence, we obtain a “quantum to classical” principle for the equivariant quantum  $K$ -theory as follows. We define (see Section 3) for each degree  $d$  a space  $Y_d$  homogeneous under the group  $\text{Aut}(X)$  and consider the incidence diagram

$$\begin{array}{ccc} Z_d & \xrightarrow{p} & X \\ q \downarrow & & \\ Y_d & & \end{array}$$

Denoting by  $I_d^T(\alpha, \beta, \gamma)$  the equivariant  $K$ -theoretic Gromov-Witten invariant and using the general framework developed in [BuMi08], we obtain the following:

**Theorem 0.3** *For  $X$  a cominuscule homogeneous space and  $d$  any non negative integer, we have the following formula:*

$$I_d^T(\alpha, \beta, \gamma) = \chi_{Y_d}(q_* p^* \alpha \cdot q_* p^* \beta \cdot q_* p^* \gamma)$$

where  $\chi_{M_d} : K_T(M_d) \rightarrow K_T(\text{pt})$  is the  $T$ -equivariant Euler characteristic and where  $\alpha, \beta$  and  $\gamma$  are three classes in  $K_T(X)$ .

We now explain our techniques to get rationality. By [ChMaPe08, Proposition 3.17, Fact 3.18], for  $d \leq d_{\max}$ , this Gromov-Witten variety is one point. On the other hand, the cases of type  $A$  have already been proved in [BuMi08]. Thus we stick to the other cases.

We have two different arguments, each of them being not enough alone to cover all the cases. On the one hand, we explain that the quadrics, the symplectic Grassmannians of type  $C$ , the spinor varieties of even type  $D_{2n}$ , and the exceptional case  $E_7/P_7$ , share some common geometric properties which imply that the restriction to  $X$  of the projection with center an osculating space is a birational morphism to a projective space, and that we can describe the set of curves in this projective space which are the projection of some curve in  $X$  (Proposition 1.5). From this follows an explicit birational description of the Gromov-Witten varieties, and in particular the fact that they are rational.

On the other hand, if the Dynkin diagram of  $X$  has a cominuscule node which is not the marked node (which occurs in all cases but the odd quadrics, the symplectic Grassmannians and  $E_7/P_7$ ), then we can apply results of [Per02] to show that a big open subset of  $X$  is the total space of a globally generated vector bundle  $E$  on a smaller homogeneous space  $Y$ . It follows that a dense open subset of the space of rational curves of degree  $d$  in  $X$  is a vector bundle over the space of rational curves of degree  $d$  in  $Y$ . We then show that on a generic rational curve of degree  $d$  in  $Y$  the restriction of  $E$  is the vector bundle  $\oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$  with  $a_i \geq 2$  for all  $i$ , which implies the rationality of the Gromov-Witten locus by induction.

At the end of Section 2, we also explain how the above vector bundle technique allows us to prove some rationality results for Gromov-Witten varieties for non cominuscule homogeneous spaces and large degrees (this question was also raised in [BuMi08]). We deal with orthogonal Grassmannians and adjoint varieties (see Proposition 2.18 and Proposition 2.19).

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# 1 The case of generalised Veronese curves

In this section we prove that Gromov-Witten varieties are rational for a class of homogeneous spaces that share common geometric properties and that we call, after Mukai [Muk98], generalised Veronese curves. These homogeneous spaces are: the quadrics, the Lagrangian Grassmannians  $\mathbb{G}_\omega(n, 2n)$ , the Grassmannians  $\mathbb{G}(n, 2n)$ , the “even” spinor varieties  $\mathbb{G}_Q(2n, 4n)$ , and  $E_7/P_7$ .

More precisely, let  $a \geq 1$  and  $n \geq 2$  be integers. Assume that  $a \in \{1, 2, 4, 8\}$  if  $n = 3$ , and that  $a \in \{1, 2, 4\}$  if  $n \geq 4$ . We define a variety  $X(a, n)$  by the following:

$n = 2$	$a$ arbitrary	$\mathbb{Q}^a$
$n \geq 3$	$a = 1$	$\mathbb{G}_\omega(n, 2n)$
$n \geq 3$	$a = 2$	$\mathbb{G}(n, 2n)$
$n \geq 3$	$a = 4$	$\mathbb{G}_Q(2n, 4n)$
$n = 3$	$a = 8$	$E_7/P_7$

**Remark 1.1** Let  $X$  be a cominuscule homogeneous space (see [ChMaPe08] for more about cominuscule homogeneous spaces). Denote  $d_{\max}$  the smallest integer such that through any two points of  $X$  there passes a rational curve of degree  $d_{\max}$ . The varieties  $X(a, n)$  are all the cominuscule homogeneous spaces  $X$  such that through any *three* points passes a rational curve of degree  $d_{\max}$ .

## 1.1 Generalised Veronese curves

To prove that  $X(n, a)$  has rational Gromov-Witten loci, we give a parametrisation of the set of curves in  $X(n, a)$  of a given degree passing through a point  $x \in X(n, a)$ . To this end our first task is to show that the projection from an osculating space to  $x$  at order  $n - 2$  is a birational map to a projective space, by interpreting  $X(n, a)$  as a generalised rational normal curve of degree  $n$ .

More precisely, let  $G(n, a)$  be the fundamental covering of the automorphism group of  $X(n, a)$ ,  $V(n, a)$  the representation of  $G(n, a)$  where  $X(n, a)$  is minimally embedded,  $L(n, a)$  the derived group of a Levi factor of the parabolic subgroup of  $G(n, a)$  stabilising a point in  $X(n, a)$ , and  $T(n, a)$  a tangent space of  $X(n, a)$ , which is a representation of  $L(n, a)$ . We display the above

defined groups and representations in the following array.

$n$	$a$	$G(n, a)$	$V(n, a)$	$L(n, a)$	$T(n, a)$
2	arbitrary	$Spin_{a+2}$	$\mathbb{C}^{a+2}$	$Spin_a$	$\mathbb{C}^a$
$n \geq 3$	1	$Sp_{2n}$	$(\wedge^n \mathbb{C}^{2n})_\omega$	$SL_n$	$S^2 \mathbb{C}^n$
$n \geq 3$	2	$SL_{2n}$	$\wedge^n \mathbb{C}^{2n}$	$SL_n \times SL_n$	$\mathbb{C}^n \otimes \mathbb{C}^n$
$n \geq 3$	4	$Spin_{4n}$	$\mathbb{S}_+$	$SL_{2n}$	$\wedge^2 \mathbb{C}^{2n}$
3	8	$E_7$	$V^{56}$	$E_6$	$V^{27}$

( $\mathbb{S}^+$  denotes a spinor representation of  $Spin_{4n}$ )

Moreover let  $Z(n, a)$  be the center of a Levi factor of a parabolic subgroup of  $G(n, a)$  stabilising a point in  $X(n, a)$ : we have  $Z(n, a) \simeq \mathbb{C}^*$ . This group induces a decomposition

$$V(n, a) = V_{-n} \oplus V_{-n+2} \oplus \cdots \oplus V_{n-2} \oplus V_n$$

as a  $L(n, a)$ -representation, where  $V_i$  is the  $i$ -th eigenspace of  $Z(n, a)$ . For example, for  $a = 8$  we have  $V^{56} = \mathbb{C} \oplus V^{27} \oplus V^{27} \oplus \mathbb{C}$ , which is known as the Zorn-matrix expression for elements in  $V^{56}$  (see for example [Gar01, Example 2.3 and Theorem 4.15]). The other cases can be checked directly. The following fact is an easy exercise on plethysm:

**Fact 1.2** *For  $2 \leq i \leq n$  the space of  $L(n, a)$ -equivariant maps of degree  $i$  from  $T(n, a)$  to  $V_{-n+2i}$  is isomorphic to  $\mathbb{C}$ .*

The choice of a specific such map does not really matter, but to fix the ideas we denote  $\nu_i$  one degree  $i$  map from  $T(n, a)$  to  $V_{-n+2i}$ . In the case  $n \geq 3$  and  $a = 2$ , we have  $T(n, a) = \mathbb{C}^n \otimes \mathbb{C}^n$  and  $V_{-n+2i} = \wedge^i \mathbb{C}^n \otimes \wedge^i \mathbb{C}^n$ , thus we denote  $\nu_i(x)$  the element given by the  $i \times i$  minors of the matrix  $x \in \mathbb{C}^n \otimes \mathbb{C}^n$ . By restriction to symmetric matrices we also define  $\nu_i$  in case  $a = 1$ . For skew-symmetric matrices (case  $a = 4$ ), we define  $\nu_i(x)$  for  $x \in \wedge^2 \mathbb{C}^{2n}$  as the collection of the Pfaffians of the symmetric  $2i \times 2i$  minors of  $x$ .

The interpretation of  $X(n, a)$  as a generalised Veronese curve of degree  $n$  comes from the following proposition which is proved in [LaMa02, Section 3.1]:

**Proposition 1.3** *Let  $v \in V(n, a)$  and denote  $v = (v_k)$  according to the above decomposition  $V(n, a) = \oplus_k V_k$ . Assume that  $v_{-n} = 1$ , then the class of  $v$  in  $\mathbb{P}V(n, a)$  belongs to  $X(n, a)$  if and only if  $v_{-n+2i} = \nu_i(v_{-n+2})$  for  $2 \leq i \leq n$ .*

*Proof.* For self-completeness we include a quick elementary proof of this result which is part of the general construction performed in [LaMa02]. When  $n = 2$  this result is obvious, and when  $n = 3$  and  $a = 8$  it is proved in [Muk98]. Thus we assume  $n \geq 3$  and  $a \in \{1, 2, 4\}$ . Let us first consider the case when  $a = 2$ . Consider a base  $(e_1, \dots, e_n, f_1, \dots, f_n)$  of  $\mathbb{C}^{2n}$  and a completely decomposable form  $v \in \wedge^n \mathbb{C}^{2n}$ , i.e. the class of  $v$  in  $\mathbb{P} \wedge^n \mathbb{C}^{2n}$  is an element of the Grassmannian  $\mathbb{G}(n, 2n)$ , and write  $v = (v_k)$  in the decomposition  $V = \oplus_k V_k$ . Denoting  $E$  resp.  $F$  the subspace of  $\mathbb{C}^{2n}$  generated by the  $e_i$ 's resp. the  $f_i$ 's, we have  $\mathbb{C}^{2n} = E \oplus F$  and  $V_{-n+2i} = \wedge^{n-i} E \otimes \wedge^i F$ . Since by assumption  $v_{-n} \neq 0$ , the coordinate of  $v \in \wedge^n \mathbb{C}^{2n}$  on  $e_1 \wedge \cdots \wedge e_n$  does not vanish, and thus  $v$  can be written as  $(e_1 + \varphi(e_1)) \wedge \cdots \wedge (e_n + \varphi(e_n))$ , where  $\varphi : E \rightarrow F$  is a linear map. The element  $v_{-n+2}$  is then

$$\sum_i e_1 \wedge \cdots \wedge \varphi(e_i) \wedge \cdots \wedge e_n;$$

this element in  $\wedge^{n-1}E \otimes F$  identifies with  $\varphi \in \text{Hom}(E, F)$ . The element  $v_{-n+4}$  is given by

$$\sum_{i < j} e_1 \wedge \cdots \wedge \varphi(e_i) \wedge \cdots \wedge \varphi(e_j) \wedge \cdots \wedge e_n :$$

it corresponds to the element in  $\text{Hom}(\wedge^2 E, \wedge^2 F)$  given by the 2 by 2 minors of  $\varphi$ , namely  $\nu_2(v_{-n+2})$ . Similarly we have  $v_{-n+2i} = \nu_i(v_{-n+2})$  for any  $i$ . Thus the proposition is proved in the case of Grassmannians.

The case  $a = 1$  follows using the injections  $S^2\mathbb{C}^n \subset \mathbb{C}^n \otimes \mathbb{C}^n$  and  $X(n, 1) \subset X(n, 2)$ . The case  $a = 4$  also follows from the injections  $\wedge^2\mathbb{C}^{2n} \subset \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$  and  $\mathbb{G}_Q(2n, 4n) \subset \mathbb{G}(2n, 4n)$ , recalling that the Plücker embedding  $\mathbb{G}_Q(2n, 4n) \subset \mathbb{P} \wedge^{2n} \mathbb{C}^{4n}$  is the second Veronese embedding of  $\mathbb{G}_Q(2n, 4n) \subset \mathbb{P}V(n, a)$ , and that the squares of the Pfaffians of the symmetric submatrices are the minors.  $\square$

## 1.2 Projection from osculating spaces

Let  $x \in \mathbb{P}V(n, a)$  resp.  $y \in \mathbb{P}V(n, a)$  be the projectivisation of the highest resp. lowest weight line, these are points in  $X(n, a)$ . Moreover we can assume that  $x$  resp.  $y$  is the class in  $\mathbb{P}V(n, a)$  of  $(1, 0, \dots, 0)$  resp.  $(0, \dots, 0, 1)$  in the decomposition  $V(n, a) = \oplus_k V_k$ . From Proposition 1.3 we get immediately the following corollary:

**Corollary 1.4** *We have  $T_x^{n-2}X = \bigoplus_{i \leq n-2} V_{-n+2i}$  and  $T_y^1X = V_{n-2} \oplus V_n$ . Moreover, the restriction to  $X$  of the projection to  $\mathbb{P}T_y^1X$  with center  $\mathbb{P}T_x^{n-2}X$  is birational.*

We denote  $Y \subset \mathbb{P}V_{n-2} \subset \mathbb{P}T_y^1X$  the closed  $L(n, a)$ -orbit in  $\mathbb{P}V_{n-2}$ . It has dimension  $(n-1)a$ . This corollary enables one to characterise the curves in  $\mathbb{P}T_y^1X$  which are projections of curves in  $X$ :

**Proposition 1.5** *A rational curve of degree  $d-n+1$  in  $\mathbb{P}T_y^1X$  is the projection of a rational curve of degree  $d$  in  $X$  passing through  $x$  if and only if it passes through  $d-n$  points in  $Y \subset \mathbb{P}T_y^1X$ .*

*Proof.* We denote  $p : \mathbb{P}V(n, a) \dashrightarrow \mathbb{P}T_y^1X$  the projection of Corollary 1.4. Let  $C \subset X$  be a rational curve of degree  $d$  passing through  $x$ ; we show that  $p(C)$  is a rational curve of degree  $d-n+1$  which passes through  $d-n$  points in  $Y$ . In fact, since  $x \in C \subset X$  and  $T_x^{n-2}X$  is the  $(n-2)$ -th osculating space, the intersection  $C \cap T_x^{n-2}X$  has multiplicity at least  $n-1$ , and generically exactly  $n-1$ , hence  $p(C)$  has degree  $d-n+1$ .

On the other hand, the intersection of  $C$  with the hyperplane  $T_x^{n-1}X$  has length  $d$ , and the multiplicity of  $x$  is  $n$ ; thus there are  $d-n$  other intersection points of  $C$  with  $T_x^{n-1}X$ . Let  $z$  be any of these points and write  $z = (z_i)$  with  $z_k \in V_k$ . If  $C$  is generic then  $z_{-n} \neq 0$ , so that we can assume that  $z_{-n} = 1$ , and Proposition 1.3 implies that  $\nu_n(z_{-n+2}) = z_n = 0$  and  $z_{n-2} = \nu_{n-1}(z_{-n+2})$ . Thus it follows that  $[z_{n-2}] \in Y$ . Thus for  $C$  a generic curve  $p(C)$  meets  $Y$  along at least  $d-n$  points, and therefore this holds for any  $C$ .

Observe now that the space of curves of degree  $d-n+1$  passing through  $d-n$  points in  $Y$  is irreducible. In fact, we can specify such a curve by choosing any  $d-n$  points on  $Y$  and then a curve of degree  $d-n+1$  in  $\mathbb{P}T_y^1X$  passing through these points. Since the space of curves of degree  $d-n+1$  through  $d-n$  fixed points in a projective space is non empty and irreducible, the claim is proved.

To finish the proof of the proposition, it is enough to show that a generic curve of degree  $d-n+1$  passing through  $d-n$  points in  $Y$  is the projection of a curve of degree  $d$  in  $X$ . To this end a dimension count argument is enough. Note that we have  $c_1(X) = an + 2 - a$  and

$\dim(X) = \frac{1}{2}(an^2 + 2n - an)$ . Thus the space of curves of degree  $d$  in  $X$  and passing through  $x$  has dimension  $d(an+2-a)-2$ . On the other hand the space of curves of degree  $d-n+1$  in  $\mathbb{P}(T_y^1)$  passing through  $d-n$  points in  $Y$  has dimension  $(d-n+1)(\dim T_y^1 X) + \dim \mathbb{P}T_y^1 X - 3 - (d-n)\text{codim}_{\mathbb{P}T_y^1 X}(Y)$ . This is equal to  $(d-n+2)(\dim X + 1) - 4 - (d-n)(n-1 + \frac{1}{2}(n-1)(n-2)a)$ . This is also  $d(an+2-a)-2$ . Thus the proposition is proved.  $\square$

**Theorem 1.6** *Let  $X = X(n, a)$  be a generalised Veronese curve and  $d$  an integer. The space of rational curves of degree  $d$  passing through 3 fixed points in  $X$  is empty for  $d < n$  and rational for  $d \geq n$ .*

*Proof.* For  $d < n$  an easy dimension count proves the result.

For  $d \geq n$ , by Proposition 1.5, a generic curve of degree  $d$  passing through 3 fixed points in  $X$  corresponds to a generic curve of degree  $d-n+1$  passing through 2 fixed points in  $\mathbb{P}T_y^1 X$  and passing moreover through  $d-n$  points in  $Y$ . Thus it is enough to show that this latter space is rational.

This in turn follows from the fact  $Y$  is rational and that in any projective space the space of rational curves of degree  $\delta$  ( $\delta = d-n+1$ ) passing through  $\delta+1$  fixed points is non empty and rational.  $\square$

## 2 Rationality by the vector bundle technique

### 2.1 Inductive birational realisation of homogeneous varieties

In this subsection, we recall some results of [Per02] that we shall use in the sequel. Let  $Q$  be a parabolic subgroup of  $G$  containing  $B$ . Let  $w_0$  be the longest element of the Weyl group. Consider the subvariety  $Z = Q^{w_0}/Q^{w_0} \cap P$  of  $G/P$  where  $Q^{w_0}$  denotes the conjugate of  $Q$  under  $w_0$ . Because  $Q^{w_0}$  contains the opposite Borel subgroup  $B^-$ , the subvariety  $Z$  is open in  $X$ . Furthermore, denoting by  $i$  the Weyl involution, one easily checks (for more details see [Per02, Proposition 6]) that the complement of  $Z$  has codimension at least 2 as soon as the sets of simple roots  $\Sigma(P)$  and  $i(\Sigma(Q))$  are disjoint.

Let us consider the Levi decomposition of  $Q$  given by  $Q = U_Q \rtimes L_Q$  where  $U_Q$  is the unipotent radical of  $Q$  and  $L_Q$  is a section of the quotient  $Q/U_Q$ . This decomposition induces a decomposition of  $Q^{w_0}$  as follows  $Q^{w_0} = U_Q^{w_0} \rtimes L_Q^{w_0}$ . Let us denote by  $Y$  the quotient  $L_Q^{w_0}/L_Q^{w_0} \cap P$ , we have a natural morphism  $p : Z \rightarrow Y$  induced by the projection of  $Q^{w_0}$  to  $L_Q^{w_0}$ . As a consequence of [Per02, Proposition 5] we have the

**Proposition 2.1** *Let  $Q$  be a maximal parabolic subgroup such that  $\Sigma(Q)$  is a cominuscule root, then the map  $p$  realises  $Z$  as the vector bundle over  $L_Q^{w_0}/L_Q^{w_0} \cap P$  defined by the representation of  $L_Q^{w_0} \cap P$  in  $U_Q^{w_0}/U_Q^{w_0} \cap P$ .*

*Proof.* This statement is contained in [Per02, Proposition 5] with the exception that  $p$  is in general an affine bundle. However, as explained at the end of the proof of [Per02, Proposition 5], this affine bundle is an actual vector bundle as soon as the unipotent subgroup  $U_Q^{w_0}$  is abelian. This is the case as soon as  $\Sigma(Q)$  is a cominuscule root.  $\square$

**Remark 2.2** As explained in [Per02, Proposition 5], the above vector bundle  $p : Z \rightarrow Y$  is globally generated.

## 2.2 Passing rationality of Gromov-Witten varieties through vector bundles

In this subsection, we let  $p : X \rightarrow Y$  be the vector bundle defined in the previous subsection and we explain how to deduce the rationality of Gromov-Witten varieties in  $Z$  from the rationality of Gromov-Witten varieties in  $Y$  under simple hypothesis on the vector bundle. Let us denote by  $E$  the locally free sheaf on  $Y$  defined by taking local sections of the vector bundle  $p : Z \rightarrow Y$ . Let us assume that the Picard group of  $Y$  and hence of  $Z$  is  $\mathbb{Z}$ .

**Proposition 2.3** *Let  $x, y$  and  $z$  be three points in general position in  $Z$  and assume that  $d$  is a non negative integer such that there exists a degree  $d$  morphism  $f : \mathbb{P}^1 \rightarrow Y$  with  $H^1(\mathbb{P}^1, f^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0$ .*

*Assume that the Gromov-Witten variety of degree  $d$  rational curves in  $Y$  passing through  $p(x), p(y)$  and  $p(z)$  is rational, then the Gromov-Witten variety of degree  $d$  rational curves in  $Z$  passing through  $x, y$  and  $z$  is also rational.*

*Proof.* The ideas for the proof are partially already contained in the proof of [Per02, Proposition 3]. Indeed, the map  $p$  induces by composition a morphism  $q : \text{Mor}_d(\mathbb{P}^1, Z) \rightarrow \text{Mor}_d(\mathbb{P}^1, Y)$  which is according to [Per02, Proposition 3] an affine bundle and in our situation a vector bundle. More precisely define the morphisms  $\text{pr}$  and  $\text{ev}$  by the following diagram

$$\begin{array}{ccc} \text{Mor}_d(\mathbb{P}^1, Y) \times \mathbb{P}^1 & \xrightarrow{\text{ev}} & Y \\ \downarrow \text{pr} & & \\ \text{Mor}_d(\mathbb{P}^1, Y) & & \end{array}$$

Then, according to the proof of [Per02, Proposition 3] the vector bundle defined by  $q$  is associated to the locally free sheaf  $\text{ev}^* \text{pr}_* E$ .

Let  $f : \mathbb{P}^1 \rightarrow Y$  be an element in  $\text{Mor}_d(\mathbb{P}^1, Y)$  and consider the vector bundle  $Z_f \rightarrow \mathbb{P}^1$  obtained by pull-back. The associated locally free sheaf is  $f^*E$ . The fibre  $q^{-1}(f)$  is the set of global sections  $H^0(\mathbb{P}^1, f^*E)$ . Now if we take  $f$  passing through  $p(x), p(y)$  and  $p(z)$ , then a section  $s$  passes through  $x, y, z$  if  $\text{ev}(s) = (e_x, e_y, e_z)$ , where  $\text{ev} : H^0(\mathbb{P}^1, f^*E) \rightarrow E_{p(x)} \times E_{p(y)} \times E_{p(z)}$  is the evaluation at  $f^{-1}(p(x)), f^{-1}(p(y)), f^{-1}(p(z))$  and  $e_x \in E_{p(x)}, e_y \in E_{p(y)}, e_z \in E_{p(z)}$  are some fixed elements determined by  $x, y, z$ . By hypothesis there exists a morphism  $f$  such that all factors of  $f^*E$  have degree at least 2. By semi-continuity this is the case for a general morphism (the space of morphisms  $\text{Mor}_d(\mathbb{P}^1, Y)$  is irreducible, see for example [Per02]). In this case  $\text{ev}$  is surjective and  $q^{-1}(f)$  is an affine subspace of  $H^0(\mathbb{P}^1, f^*E)$ . The result follows.  $\square$

To prove rationality results with the vector bundle technique, it is therefore enough to produce morphisms  $f : \mathbb{P}^1 \rightarrow Y$  such that the cohomology group  $H^1(\mathbb{P}^1, f^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3))$  vanishes. Using the following lemma, we will only need to prove this for small degree morphisms.

**Lemma 2.4** *Assume that for a general morphism  $f : \mathbb{P}^1 \rightarrow Y$  of degree  $d$ , the cohomology group  $H^1(\mathbb{P}^1, f^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3))$  vanishes. Then for any  $d' \geq d$ , and a general morphism  $g : \mathbb{P}^1 \rightarrow Y$  of degree  $d'$ , the cohomology group  $H^1(\mathbb{P}^1, g^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3))$  vanishes.*

*Proof.* Let  $f : \mathbb{P}^1 \rightarrow Y$  a degree  $d$  morphism such that the cohomology group  $H^1(\mathbb{P}^1, f^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3))$  vanishes. Denote  $y_1$  resp.  $y_2, y_3$  the element  $f(0)$  resp.  $f(1), f(\infty)$ . Let  $g$  be a stable morphism  $\mathbb{P}^1 \cup \mathbb{P}^1 \rightarrow Y$  of degree  $d'$  obtained as the map  $f$  on the first factor and a map  $f' : \mathbb{P}^1 \rightarrow Y$  of degree  $d' - d$  whose image is not passing through  $y_1, y_2$  or  $y_3$  but meets  $f(\mathbb{P}^1)$  in one point say  $y_4$ . For

the curve  $g(\mathbb{P}^1 \cup \mathbb{P}^1)$  we have the exact sequence  $0 \rightarrow f'^*(E(-y_4)) \rightarrow g^*E \rightarrow f^*E \rightarrow 0$  and because the points  $y_1, y_2$  and  $y_3$  are not on  $f'(\mathbb{P}^1)$  we get the exact sequence

$$0 \rightarrow f'^*(E(-y_4)) \rightarrow g^*(E(-y_1 - y_2 - y_3)) \rightarrow f^*(E(-y_1 - y_2 - y_3)) \rightarrow 0.$$

Because  $E$  is globally generated (see Remark 2.2), we get the vanishing of  $H^1(\mathbb{P}^1, f'^*(E(-y_4))) = 0$  and by hypothesis we have  $H^1(\mathbb{P}^1, f^*(E(-y_1 - y_2 - y_3))) = H^1(\mathbb{P}^1, f^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0$ . We deduce the condition  $H^1(\mathbb{P}^1 \cup \mathbb{P}^1, g^*(E(-y_1 - y_2 - y_3))) = 0$ . By semi-continuity the same holds for a general degree  $d'$  stable map  $g : \mathbb{P}^1 \rightarrow Y$  with three marked points  $0, 1$  and  $\infty$  *i.e.* we have  $H^1(\mathbb{P}^1, g^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0$ . The result follows. Remark that we used the irreducibility of the space of space maps in  $Y$  (see for example [Tho98]).  $\square$

## 2.3 Proof of the rationality

In this subsection we prove the rationality of certain Gromov-Witten varieties using Proposition 2.1 and 2.3. We shall deal with the following homogeneous spaces  $X$ : the Grassmannians  $\mathbb{G}_Q(n, 2n)$  of maximal isotropic subspaces in an even dimension vector space with a non degenerate quadratic form and the Cayley plane  $E_6/P_1$ . We denote by  $\text{GW}_d(X)$  the Gromov-Witten variety of degree  $d$  curves passing through three general points in  $X$ .

### 2.3.1 Orthogonal Grassmannians

Let us start with  $X = \mathbb{G}_Q(n, 2n) = G/P$  with  $P$  the maximal parabolic subgroup with  $\Sigma(P) = \{\alpha_n\}$  with notation as in [Bou54]. Take  $Q$  to be the maximal parabolic subgroup such that  $\Sigma(P) \cup i(\Sigma(Q)) = \{\alpha_{n-1}, \alpha_n\}$ . The open subset  $Z$  of  $X$  defined by  $Q$  as above has complementary of codimension at least two since  $i(\Sigma(Q)) \cap \Sigma(P)$  is empty. Furthermore, by Proposition 2.1, the morphism  $p : Z \rightarrow Y$  defined above is a vector bundle. The variety  $Y$  is easily seen to be isomorphic to  $\mathbb{P}^{n-1}$  and the locally free sheaf  $E$  associated to  $p$  is  $\Lambda^2 T_{\mathbb{P}^{n-1}}(-1)$ . We conclude by Proposition 2.3 for  $d \geq n - 1$  using the following

**Proposition 2.5** *For  $d \geq n - 1$ , there exists a degree  $d$  morphism  $f : \mathbb{P}^1 \rightarrow Y$  such that the cohomology group  $H^1(\mathbb{P}^1, f^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3))$  vanishes.*

*Proof.* By Lemma 2.4, we only need to construct a degree  $n - 1$  morphism  $f : \mathbb{P}^1 \rightarrow Y$  with  $H^1(\mathbb{P}^1, f^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = 0$ . But for a general degree  $n - 1$  morphism  $f : \mathbb{P}^1 \rightarrow Y$  we have  $f^*(T_{\mathbb{P}^{n-1}}(-1)) = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}$  and the result follows by taking the second exterior power.  $\square$

Using the fact that the Gromov-Witten variety of curves passing through three points in  $\mathbb{P}^n$  is rational and Proposition 2.3, we obtain:

**Corollary 2.6** *The Gromov-Witten varieties of degree  $d$  curves passing through three general points in  $G_Q(n, 2n)$  is rational for  $d \geq n - 1$ .*

For  $d < n/2$ , we know by dimension counting, that there is no degree  $d$  curve passing through three general points in  $G_Q(n, 2n)$ . For  $n$  even and  $d = n/2$ , then by a result of A. Buch, A. Kresch and A. Tamvakis [BuKrTa03], there is a unique degree  $d$  curve passing through 3 general points in  $G_Q(n, 2n)$ . We are left to deal with curves of degree  $d$  with  $n/2 < d < n - 1$ . For this we will describe more precisely the geometry of the curves through three fixed points. We first make the following remark:



**Remark 2.7** Let  $F$  be a proper closed subset of  $\text{Mor}_d(\mathbb{P}^1, X)$ . Given  $x, y, z \in X$  three generic points, denote by  $\text{Mor}_d(\mathbb{P}^1, X, \{x, y, z\})$  the subscheme of morphisms  $f$  such that  $f(0) = x, f(1) = y, f(\infty) = z$ , and assume it is not empty. By Kleiman's transversality Theorem [Kle74], the scheme  $\text{Mor}_d(\mathbb{P}^1, X, \{x, y, z\})$  is equidimensional of dimension  $dc_1(X) - 2 \dim X$  and  $\text{Mor}_d(\mathbb{P}^1, X, \{x, y, z\}) \cap F$  is empty or equidimensional of dimension  $dc_1(X) - 2 \dim X - \text{codim } F$ . Therefore in any irreducible component of  $\text{Mor}_d(\mathbb{P}^1, X, \{x, y, z\})$  there exist elements outside  $F$ .

Using this remark and because we want to prove the rationality of the Gromov-Witten variety, we will only need to study general degree  $d$  morphisms. In particular, we may assume that for  $f$  such a morphism the decomposition of  $f^*K$ , where  $K$  is the tautological subbundle in  $X$ , is the general decomposition. This decomposition is given in the following:

**Lemma 2.8** *Let  $K$  be the tautological subbundle on  $\mathbb{G}_Q(n, 2n)$ , then for a general morphism  $f : \mathbb{P}^1 \rightarrow \mathbb{G}_Q(n, 2n)$  of degree  $d = n - a$  we have  $f^*K = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2a} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus n-2a}$ .*

*Proof.* To prove this result, we decompose the vector space  $\mathbb{C}^{2n}$  as a direct sum  $\oplus_i V_i$  of mutually orthogonal vector spaces such that the restriction of the quadratic form is non degenerate on  $V_i$  and such that  $V_i$  as dimension  $2n_i$  equal to 4 or 6. We remark that we may define an embedding:

$$\prod_i \mathbb{G}_Q(n_i, 2n_i) \rightarrow \mathbb{G}_Q(n, 2n)$$

by taking the direct sum of the subspaces in each subspace  $V_i$ . The pull-back of the tautological subbundle is the direct sum of the tautological subbundles of each factor  $\mathbb{G}_Q(n_i, 2n_i)$ . Now because  $\mathbb{G}_Q(n_i, 2n_i)$  is isomorphic to  $\mathbb{P}^1$  or  $\mathbb{P}^3$  and the tautological subbundle  $K_i$  is  $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$  or  $\wedge^2(\Omega_{\mathbb{P}^3}^1(1))$ , we know the decomposition on  $\mathbb{P}^1$  of the pull-back of a degree  $d_i$  morphism  $f_i : \mathbb{P}^1 \rightarrow \mathbb{G}_Q(n_i, 2n_i)$ . In particular, we may assume that  $f^*K_i$  has only factors of the form  $\mathcal{O}_{\mathbb{P}^1}(-1)$  or  $\mathcal{O}_{\mathbb{P}^1}(-2)$  and the result follows from the fact that the degree of  $K$  on  $\mathbb{G}_Q(n, 2n)$  is -2.  $\square$

**Proposition 2.9** *For  $n/2 < d < n-1$  and  $x, y$  and  $z$  three general points in  $\mathbb{G}_Q(n, 2n)$ , the variety of degree  $d$  morphisms  $f : \mathbb{P}^1 \rightarrow \mathbb{G}_Q(n, 2n)$  with  $f(0) = x, f(1) = y$  and  $f(\infty) = z$  is rational.*

*Proof.* Let us set  $d = n - a$ . By the previous Lemma we know that, for a fixed non degenerate quadratic form on  $\mathbb{C}^{4a}$  (resp. on  $\mathbb{C}^{2n-4a}$ ), there exists a map  $\mathcal{O}_{\mathbb{P}^1}(-1)^{2a} \rightarrow \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{4a}$  (resp.  $\mathcal{O}_{\mathbb{P}^1}(-2)^{n-2a} \rightarrow \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{2n-4a}$ ) defining a degree  $a$  (resp.  $n - 2a$ ) morphism  $f_1 : \mathbb{P}^1 \rightarrow \mathbb{G}_Q(2a, 4a)$  (resp.  $f_2 : \mathbb{P}^1 \rightarrow \mathbb{G}_Q(n - 2a, 2n - 4a)$ ). Taking  $\mathbb{C}^{2n}$  as the orthogonal direct sum of  $\mathbb{C}^{4a}$  and  $\mathbb{C}^{2n-4a}$  we obtain a map  $\mathcal{O}_{\mathbb{P}^1}(-1)^{2a} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{n-2a} \rightarrow \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{2n}$  defining a degree  $n - a$  morphism  $f : \mathbb{P}^1 \rightarrow \mathbb{G}_Q(n, 2n)$  and such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^1}(-1)^{2a} & \xrightarrow{\quad} & \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{4a} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{P}^1}(-1)^{2a} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{n-2a} & \xrightarrow{\quad} & \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{2n} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{P}^1}(-2)^{n-2a} & \xrightarrow{\quad} & \mathcal{O}_{\mathbb{P}^1} \otimes \mathbb{C}^{2n-4a}. \end{array}$$

By Remark 2.7, we may assume that a general  $f$  in the Gromov-Witten variety satisfies the above conditions. In particular, such a map  $f$  defines a  $4a$ -dimensional subspace  $\mathbb{C}^{4a}$  of  $\mathbb{C}^{2n}$  such

that the restriction of the quadratic form is non degenerate and any subspace  $f(t)$  meets  $\mathbb{C}^{4a}$  in dimension  $2a$ . In particular  $\mathbb{C}^{4a}$  meets the three subspaces corresponding to  $x$ ,  $y$  and  $z$  in dimension  $2a$  defining therefore three points  $x'$ ,  $y'$  and  $z'$  in  $\mathbb{G}_Q(2a, 4a)$ . The map  $f$  induces a degree  $a$  morphism  $f_1 : \mathbb{P}^1 \rightarrow \mathbb{G}_Q(2a, 4a)$  passing through  $x'$ ,  $y'$  and  $z'$ . We already know that there is a unique such morphism. By projection with respect to  $\mathbb{C}^{4a}$  the morphism  $f$  defines a degree  $n - 2a$  morphism  $f_2 : \mathbb{P}^1 \rightarrow \mathbb{G}_Q(n - 2a, 2n - 4a)$  passing through the images, say  $x''$ ,  $y''$  and  $z''$ , of  $x$ ,  $y$  and  $z$  under the projection. By Corollary 2.6 (here the degree is  $D = n - 2a \geq N - 1 = n - 2a - 1$  in  $G_Q(N, 2N)$ ), we know that the variety of such maps is rational. We are therefore left to prove that the variety  $\mathcal{W}$  of  $4a$ -dimensional subspaces  $W$  of  $\mathbb{C}^{2n}$  meeting the three points  $x$ ,  $y$  and  $z$  in dimension  $2a$  is rational.

For this, we need to discuss on the parity of  $n$ . If  $n$  is even, then the variety  $\mathcal{W}$  is birational to the Grassmannian  $\mathbb{G}(2a, n)$ . Indeed, if a  $4a$ -dimensional subspace  $W$  is given, its intersection with  $x$  defines a subspace  $x'$  of dimension  $2a$  in  $x$  (which is of dimension  $n$ ). Conversely, if we have a subspace  $x'$  of dimension  $2a$  in  $x$ , then by projection with respect to  $x'$ , we send  $y$  and  $z$  to subspaces of dimension  $n$  (this is because as  $n$  is even, the space  $x$ , and therefore  $x'$ , does not meet  $y$  and  $z$ ) in  $\mathbb{C}^{2n}/x'$  which is of dimension  $2n - 2a$ . The images of  $y$  and  $z$  meet in dimension  $2a$  and taking the inverse image of this subspace we obtain the desired  $4a$ -dimensional subspace.

If  $n$  is odd, we first remark that  $P = x \cap y$ ,  $Q = x \cap z$  and  $R = y \cap z$  are of dimension one. Denote by  $U$  the isotropic 3-dimensional subspace generated by  $P$ ,  $Q$  and  $R$ . Let us first prove that  $P$ ,  $Q$  and  $R$  are contained in  $W$  therefore  $U \subset W$ . Indeed let, as above, be  $x'$  the  $2a$ -dimensional intersection  $x \cap W$  and consider the projection  $p_{x'}$  by  $x'$ . Then  $p_{x'}(W)$  is a  $2a$ -dimensional subspace and  $p_{x'}(y)$  (resp.  $p_{x'}(z)$ ) is a subspace of dimension  $n - 1$  or  $n$  according to the fact that  $P$  (resp.  $Q$ ) is contained in  $x'$  or not. If we define  $y' = y \cap W$  and  $z' = z \cap W$ , then  $p_{x'}(y')$  (resp.  $p_{x'}(z')$ ) is a subspace of dimension  $2a - 1$  or  $2a$  according to the fact that  $P$  (resp.  $Q$ ) is contained in  $x'$  or not. An easy dimension count shows that we have  $p_{x'}(y) \cap p_{x'}(z) = p_{x'}(y') \cap p_{x'}(z')$  and is therefore contained in  $p_{x'}(W)$ . In particular we have  $p_{x'}(R) \in p_{x'}(W)$ , therefore  $R$  is in  $W$ . By symmetry the result follows.

Let us now consider  $(W \cap U^\perp)/U \subset (\mathbb{C}^{2n} \cap U^\perp)/U$ . These spaces are of dimension  $4a - 6$  and  $2n - 6$  and the spaces  $(x \cap U^\perp)/U$ ,  $(y \cap U^\perp)/U$  and  $(z \cap U^\perp)/U$  are of dimension  $n - 3$ . Therefore to determine  $(W \cap U^\perp)/U$  we only have to apply the even case. Now we need to determine  $W$  containing  $W \cap U^\perp$ . For this we project with respect to  $W \cap U^\perp$ . The space  $W$  becomes a 3-dimensional subspace of  $\mathbb{C}^{2n}/W \cap U^\perp$  (which is of dimension  $2n + 3 - 4a$ ) and the image of  $W$  has to meet the image of  $x$ , of  $y$  and of  $z$  in dimension 1. These images are the subspaces  $x/(x \cap W \cap U^\perp)$ ,  $y/(y \cap W \cap U^\perp)$  and  $z/(z \cap W \cap U^\perp)$  and are of dimension  $n + 1 - 2a$ . To determine  $W$  we only need to pick one point in each of the spaces  $\mathbb{P}(x/(x \cap W \cap U^\perp))$ ,  $\mathbb{P}(y/(y \cap W \cap U^\perp))$  and  $\mathbb{P}(z/(z \cap W \cap U^\perp))$  and take the inverse image by the projection with respect to  $W \cap U^\perp$  of the 3-dimensional subspace obtained as the span of these 3 points.  $\square$

### 2.3.2 The Cayley plane

In this subsection  $X$  is the Cayley plane, that is the variety  $G/P$  with  $G$  of type  $E_6$  and  $\Sigma(P) = \{\alpha_1\}$  (again with notation as in [Bou54]). Let us take  $Q = P$ . We have  $\Sigma(P) \cap i(\Sigma(P)) = \emptyset$ . With the notation of Section 2.1, we have a vector bundle  $p : Z \rightarrow Y$ . Let us denote by  $E$  the locally free sheaf on  $Y$  corresponding to this vector bundle.

**Lemma 2.10** *We have  $Y = \mathbb{Q}^8$  and  $E$  is the spinor bundle on  $Y$ .*

By a spinor bundle on a smooth quadric  $\mathbb{Q}^{2d}$  we mean the vector bundle given by a spinor representation of  $Spin_{2d-2}$ , the semi-simple quotient of the parabolic subgroup of  $Spin_{2d}$  stabilising a

point in  $\mathbb{Q}^{2d}$ .

*Proof.* The fact that  $Y$  is a 8-dimensional quadric follows from the definition of  $Y$  and the fact that the Levi subgroup  $L_P^{w_0}$  of  $P^{w_0}$  (resp.  $L_P^{w_0} \cap P$ ) is isomorphic to a group of type  $D_5$  (resp. to a parabolic subgroup of  $L_P^{w_0}$  corresponding to the root  $\alpha_1$  in  $D_5$ ).

For the vector bundle  $E$ , we know that it is given by the representation of  $L \cap P$  on  $U_P^{w_0}/(U_P^{w_0} \cap P)$  where  $U_P^{w_0}$  is the unipotent radical in  $P^{w_0}$ . This vector space has for weight under the maximal torus of  $G$  (of type  $E_6$ ) the highest root of  $G$ . Therefore it has the weight  $\varpi_5$  for the group  $L_P^{w_0}$  of type  $D_5$ . By dimension count  $U_P^{w_0}/(U_P^{w_0} \cap P)$  has to be the irreducible representation of highest weight  $\varpi_5$  and the result follows.  $\square$

**Lemma 2.11** *Let  $Q$  be a smooth quadric and  $E \rightarrow Q$  a spinor bundle. For  $d \geq 4$ , there exists a degree  $d$  morphism  $f : \mathbb{P}^1 \rightarrow Q$  such that the cohomology group  $H^1(\mathbb{P}^1, f^*E \otimes \mathcal{O}_{\mathbb{P}^1}(-3))$  vanishes.*

*Proof.* Let us first consider the case when  $Q$  is even-dimensional and denote by  $2\delta$  its dimension. By Lemma 2.4, we only need to prove this result for  $d = 4$ . For this we will prove that for a general degree 2 morphism  $f : \mathbb{P}^1 \rightarrow \mathbb{Q}^{2\delta}$ , we have  $f^*E = \mathcal{O}_{\mathbb{P}^1}(1)^{2^{\delta-1}}$ , and take its double. Consider in the root system of  $Spin_{2\delta}$  the two orthogonal roots  $\theta$  and  $\alpha_1$  where  $\theta$  is the highest root. We set  $x = x_\theta + x_{\alpha_1}$  where for  $\alpha$  root of  $Spin_{2\delta}$ , the element  $x_\alpha$  is a non zero element in the root space corresponding to  $\alpha$ . Take  $y = y_\theta + y_{\alpha_1}$  where for  $\alpha$  a root of  $Spin_{2\delta}$ ,  $y_\alpha$  is the element with weight  $-\alpha$  and such that  $[x_\alpha, y_\alpha] = \alpha^\vee$ . For  $h = [x, y] = \theta^\vee + \alpha_1^\vee$ , the triple  $(x, h, y)$  is a  $\mathfrak{sl}_2$ -triple. Denote by  $SL_2(\theta + \alpha_1)$  the associated subgroup of  $Spin_{2\delta}$ . The orbit of the projectivisation of the highest weight line under  $SL_2(\theta + \alpha_1)$  is a degree 2 curve and, since  $\langle \theta^\vee + \alpha_1^\vee, \alpha_2 \rangle = 0$ , all the weights of the spinor representation evaluated on  $\theta^\vee + \alpha_1^\vee$  are equal to one. The result follows.

For odd-dimensional quadrics, we may either reproduce a similar argument or use the facts that the restriction of a spinor bundle on  $Q^{2\delta}$  to  $Q^{2\delta-1}$  is a spinor bundle on  $Q^{2\delta-1}$  and that a rational curve of degree 4 in a smooth quadric is included in a smooth quadric of dimension 3.  $\square$

By the previous two Lemmas and Proposition 2.3 we get:

**Proposition 2.12** *For  $d \geq 4$  and  $x, y$  and  $z$  three general points in  $E_6/P_1$ , the variety of degree  $d$  morphisms  $f : \mathbb{P}^1 \rightarrow E_6/P_1$  with  $f(0) = x, f(1) = y$  and  $f(\infty) = z$  is rational.*

We are left with dealing with morphisms of degree  $d \leq 3$ . We prove the

**Proposition 2.13** *There is no curve of degree 3 passing through 3 general points in  $X$ .*

*Proof.* If such a curve exists, then the variety of all these curves has dimension 4. In particular, by the same proof as Proposition 3.2 in [Deb01], there exists a non irreducible degree 3 rational curve through the same three general points. This is not possible by the following Lemma 2.14.  $\square$

**Lemma 2.14** *There is no reducible curve of degree 3 in  $X$  passing through 3 general points.*

*Proof.* Let  $x, y, z$  be the three general points. Up to a permutation, we can assume that there is a curve  $C$  of degree 2 through  $x$  and  $y$  and a line  $l$  through  $z$  meeting  $C$ . To get a contradiction we consider the representation  $V$  of  $E_6$  in which  $X$  is embedded ( $\dim V = 27$ ). Recall that  $V$  identifies with the space of order three hermitian matrices with octonionic coordinates and  $X$  with the variety of matrices with rank one (see [Cha06]). We show that for the particular choice

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the existence of  $C$  and  $l$  leads to a contradiction.

In fact, the union of all conics through  $x$  and  $y$  is the quadric lying in the linear subspace given by the matrices of the form  $A$  below, thus  $C$  is included in this space. On the other hand  $l$  must be included in the tangent space at  $X$  to  $z$ , which is the linear subspace of the matrices of the form  $B$  below:

$$A = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}.$$

Thus  $l$  and  $C$  do not meet.  $\square$

We will however prove that there is a quantum to classical  $K$ -theoretic principle showing the following results: for a general curve  $C$  of degree 3 in  $E_6/P_1$ , there is a unique element  $\mu \in E_6/P_4$  such that  $C$  is included in the 6-dimensional Schubert variety  $S_\mu$  corresponding to  $\mu$ . There exists a curve of degree 3 passing through 3 generic points in  $S_\mu$  and the space of such curves is rational.

Let us first recall some facts about  $X = E_6/P_1$ . The following results were first proved by [Zak93] with geometric arguments; they are easy to prove once we know that there are three  $E_6$ -orbits in  $X \times X$ , which is proved for example in [ChMaPe08, Proposition 3.16]. First of all a point  $\lambda \in E_6/P_6$  defines a hyperplane section  $H_\lambda$  of  $X$ , singular along a smooth 8-dimensional quadric that we denote  $\mathbb{O}\mathbb{P}_\lambda^1$ . There are three  $E_6$ -orbits in  $X^2$ : the pairs  $(x, y)$  such that  $x = y$ , the pairs such that the line through  $x$  and  $y$  in  $\mathbb{P}V$  is included in  $X$  (or equivalently  $y \in T_x X$ ), or the generic pairs. If  $(x, y)$  is a generic pair then there is a unique  $\lambda = \lambda(x, y)$  such that  $x, y \in \mathbb{O}\mathbb{P}_\lambda^1$ . When  $x$  is fixed, the map  $y \mapsto \lambda(x, y)$ , with image the smooth 8-dimensional quadric of elements  $\lambda \in E_6/P_6$  such that  $x \in \mathbb{O}\mathbb{P}_\lambda^1$ , identifies with the projection  $X \dashrightarrow \mathbb{Q}^8$  with center  $T_x X$ . We denote  $\mathbb{O}\mathbb{P}^1(x, y) := \mathbb{O}\mathbb{P}_{\lambda(x, y)}^1$ .

Similarly there are three  $E_6$ -orbits in  $(E_6/P_6)^2$ : for generic  $(\lambda, \lambda')$  the intersection  $\mathbb{O}\mathbb{P}_\lambda^1 \cap \mathbb{O}\mathbb{P}_{\lambda'}^1$  is transverse and is one point, and for special  $(\lambda, \lambda')$ , with  $\lambda \neq \lambda'$ , this intersection is a maximal isotropic subspace in both quadrics  $\mathbb{O}\mathbb{P}_\lambda^1$  and  $\mathbb{O}\mathbb{P}_{\lambda'}^1$ ; thus it is isomorphic to  $\mathbb{P}^4$ .

Let us now explain how a general curve  $C$  of degree 3 in  $X$  determines an element  $\mu(C)$  in  $E_6/P_4$ . Let  $x, y \in C$  be general points. These are general points in  $X$ . Thus they define an element  $\lambda(x, y) \in E_6/P_6$ . In this way we get a rational map  $\mathbb{P}^2 = S^2\mathbb{P}^1 = S^2C \dashrightarrow E_6/P_6$ . In fact this rational map has degree one, as one can see by restricting the lines in  $\mathbb{P}^2 = S^2\mathbb{P}^1$  where one element is fixed, say  $x = x_0$ . In fact, the elements  $\lambda(x_0, y)$  for  $y \in C$  are the elements of the image of  $C$  under the projection with center  $T_{x_0} X$ , which is a curve of degree 1. It follows that this rational map is in fact a linear embedding of  $\mathbb{P}^2$ , and  $C$  determines a  $\mathbb{P}^2$  in  $E_6/P_6$ , or an element  $\mu(C) \in E_6/P_4$ .

For a given  $\mu \in E_6/P_4$ , there are four Schubert varieties homogeneous under the stabiliser of  $\mu$  (a parabolic subgroup conjugated to  $P_4$ ); these Schubert varieties have dimension 13, 10, 6, 2, and we denote them accordingly  $S_\mu^{13}, S_\mu^{10}, S_\mu^6, S_\mu^2$ . Each smaller Schubert variety is the singular locus of the bigger one. Remark that  $\mu$  determines also a plane in  $E_6/P_1$ : this is  $S_\mu^2$ . The variety  $S_\mu^{13}$  is the intersection of all the  $H_\lambda$  for  $\lambda \in \mathbb{P}_\mu^2 \subset E_6/P_6$ . Similarly,  $S_\mu^{10}$  is the union of the  $\mathbb{O}\mathbb{P}_\lambda^1$  for  $\lambda \in \mathbb{P}_\mu^2 \subset E_6/P_6$ . The variety  $S_\mu^6$  is the singular locus of  $S_\mu^{10}$ .

**Lemma 2.15** *We have the inclusion  $C \subset S_{\mu(C)}^6$ .*

Note that we cannot have  $C \subset S_\mu^2$  since otherwise  $C$  would only span a  $\mathbb{P}^2$ .

*Proof.* Let  $\mu = \mu(C)$  and let  $x \in C$ . By definition of  $\mu$ , if  $y \in C$  then  $\lambda(x, y) \in \mathbb{P}_\mu^2$  and therefore  $\mathbb{O}\mathbb{P}^1(x, y) \subset S_\mu^{10}$ . Thus in particular  $x \in S_\mu^{10}$  and  $C \subset S_\mu^{10}$ . But let us see that moreover  $x$  lies in

the singular locus of  $S_\mu^{10}$ . In fact since  $\mathbb{OP}^1(x, y) \subset S_\mu^{10}$  we deduce  $T_x \mathbb{OP}^1(x, y) \subset T_x S_\mu^{10}$ . For  $y' \in C$  another element, we also have  $T_x \mathbb{OP}^1(x, y') \subset T_x S_\mu^{10}$ . Moreover  $T_x \mathbb{OP}^1(x, y)$  and  $T_x \mathbb{OP}^1(x, y')$  meet along  $T_x(\mathbb{OP}^1(x, y) \cap \mathbb{OP}^1(x, y'))$  and  $\mathbb{OP}^1(x, y) \cap \mathbb{OP}^1(x, y')$  is a linear space of dimension 4, thus the linear span of  $T_x \mathbb{OP}^1(x, y)$  and  $T_x \mathbb{OP}^1(x, y')$  has dimension 12. Thus  $x$  is a singular point of  $S_\mu^{10}$ .  $\square$

In the following we denote  $S_\mu^6$  simply by  $S_\mu$  and we denote  $w = s_2 s_6 s_5 s_4 s_3 s_1$ . The Picard group of  $S_\mu$  has rank 1; however, the group of Weil divisors has rank two. Recall that a basis of the group of Weil divisors is given by the Schubert divisors. We denote  $D_6 = X(s_6 w)$  the singular one and  $D_2 = X(s_2 w)$  the divisor isomorphic to  $\mathbb{P}^5$ . Therefore, for a curve in  $S_\mu$  not meeting the singular locus of  $S_\mu$ , we may define a bidegree  $(a, b)$  by intersecting with the Weil divisors  $D_6$  and  $D_2$ . The total degree of the curve is  $a + b$  since the ample generator of the Picard group is  $D_6 + D_2$  (see for example [Per07] for more on Cartier and Weil divisors on minuscule Schubert varieties).

**Lemma 2.16** *A generic degree 3 curve  $C$  in  $X$  does not meet the singular locus  $S_{\mu(C)}^2$  of  $S_{\mu(C)}^6$  and has bidegree  $(1, 2)$ . The element  $\mu(C)$  is the only element  $\mu$  such that  $C \subset S_\mu$ .*

*Proof.* Let us first apply few facts on the space of morphisms from  $\mathbb{P}^1$  to a minuscule Schubert variety as explained in [Per05] to  $S_\mu$ . The irreducible components of the scheme of morphisms from  $\mathbb{P}^1$  to  $S_\mu$  are indexed by the pairs  $(a, b)$  of integers such that  $a + b = d$ : indeed a generic curve in the component indexed by  $(a, b)$  does not meet the singular locus of  $S_\mu$  and has bidegree  $(a, b)$ .

Let us now consider the incidence variety  $I$  of couples  $(C, \mu)$  where  $\mu \in E_6/P_4$  and  $C \subset S_\mu$  is a curve of degree 3. The projection on the second factor  $I \rightarrow E_6/P_4$  is locally trivial and has fibers above  $\mu$  given by the degree 3 curves in  $S_\mu$ . Thus there are four irreducible components in  $I$ , according to the bidegree of  $C$  in  $S_\mu$ ; we denote  $I_{(a,b)}$  (where  $a + b = 3$ ) these components. According to [Per05], the canonical sheaf on  $S_\mu$  is  $5D_2 + 6D_6$  and the space of curves of bidegree  $(a, b)$  on  $S_\mu$  has dimension  $5a + 6b + 6 = 21 + b$ , thus  $I_{(a,b)}$  has dimension  $50 + b$ .

The set  $M$  of couples of the form  $(C, \mu(C))$  with  $C$  a generic curve of degree 3 is the graph of the rational map  $\mu$  from  $\text{Mor}_3(\mathbb{P}^1, X)$  to  $E_6/P_4$  and thus irreducible. It is included in  $I$ , by Lemma 2.15. Let  $(a, b)$  be a pair of integers such that  $M \subset I_{(a,b)}$ . We first prove that  $(a, b) \neq (0, 3)$ . Indeed, this would mean that  $C$  is contained in the divisor  $D_2$ , which is a  $\mathbb{P}^5$ . This would imply that the span of  $C$  is included in  $X$ , which is not the case for a generic curve  $C$ .

Since  $M \subset I_{(a,b)}$ , we have  $52 \leq 50 + b$ , thus  $(a, b) = (1, 2)$ . By irreducibility of  $I_{(1,2)}$ ,  $M$  is dense in  $I_{(1,2)}$ , thus  $C \mapsto (C, \mu(C))$  is a rational section of the projection  $p : I_{(1,2)} \rightarrow \text{Mor}_3(\mathbb{P}^1, X)$ , which says that for a generic curve  $C$ ,  $\mu(C)$  is the only element  $\mu$  such that  $C \subset S_\mu$ , and implies that  $C$  does not meet the singular locus of  $S_{\mu(C)}$ .  $\square$

We now show that the Gromov-Witten variety of  $S_\mu$  is rational:

**Lemma 2.17** *Let  $\mu \in E_6/P_4$  and let  $x_1, x_2, x_3$  be three generic points in  $S_\mu$ . Then the space of curves of bidegree  $(1, 2)$  in  $S_\mu$  which pass through  $x_1, x_2$  and  $x_3$  is rational.*

*Proof.* We use the technique developped in Subsection 2.1. We may assume that  $\mu$  is the  $B$ -stable point in  $E_6/P_4$ , thus that  $S_\mu$  admits a  $P_4$ -action. It follows that  $S_\mu$  contains the open subset  $P_4^w / (P_1 \cap P_4^w)$ , where  $w$  denotes the element  $s_1 s_3 s_4 s_5 s_6 s_2$ . This subset admits a morphism to  $P_4^w / \langle P_1 \cap P_4^w, R_u(P_4^w) \rangle$ , which is also  $L_4^w / (P_1 \cap L_4^w)$ .

Now  $P_4$  is stable under  $s_2, s_6, s_5$ , thus we have  $P_4^w = P_4^{s_1 s_3 s_4}$ . The roots of  $P_4^w$  resp.  $L_4^w$  are thus of the form

$$\beta = s_1 s_3 s_4(\alpha) = s_1 s_3 s_4 \begin{pmatrix} a & b & c & d & e \\ & f & & & \end{pmatrix} = \begin{pmatrix} d + f - c & a + d + f - c & b + d + f - c & d & e \\ & f & & & \end{pmatrix},$$

with  $\alpha$  a root and  $c \geq 0$  resp.  $c = 0$ .

Let us compute the roots of  $P_1 \cap L_4^w$ . In this case we have  $c = 0$ , and, since  $\alpha$  is root, either  $f = 0$  or  $d = 0$ . Moreover since  $\beta$  must be a root of  $P_1$  we have  $d + f \geq 0$ , thus  $d \geq 0$  and  $f \geq 0$ . It follows that  $L_4^w/(P_1 \cap L_4^w)$  is isomorphic to  $(S_1 \times S_2 \times S_3)/(S_1 \times P_2 \times P_3)$ , with  $S_1, S_2$  resp.  $S_3$  isomorphic to  $SL_3, SL_3$  resp.  $SL_2$ , and  $P_2$  resp.  $P_3$  parabolic subgroups of  $S_2$  resp.  $S_3$  stabilising a line in  $\mathbb{C}^2$  resp.  $\mathbb{C}^3$ . Therefore  $L_4^w/(P_1 \cap L_4^w)$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^2$ .

Let us now compute the roots in  $R_u(P_4^w)/(P_1 \cap R_u(P_4^w))$ . Let  $\beta$  be such a root, and write again  $\beta = s_1 s_3 s_4(\alpha)$  with  $\alpha$  a root of  $R_u(P_4)$ . Since  $\alpha$  is a positive root, we have  $d \geq 0$  and  $f \geq 0$ . Since  $\beta$  is not a root of  $P_1$ , it is a negative root and thus  $d \leq 0$  and  $f \leq 0$ . We have  $d = f = 0$ ; since  $c > 0$  ( $\alpha \in R_u(P_4)$ ), it follows that  $c = 1$ , and  $\alpha$  is one of the roots

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & & & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & & & & \end{pmatrix}.$$

It follows that the fibration  $P_4^w/(P_1 \cap P_4^w) \rightarrow P_4^w/\langle P_1 \cap P_4^w, R_u(P_4^w) \rangle$  is isomorphic to the total space of the vector bundle  $V_1 \otimes \mathcal{O}(1, 1) \rightarrow (S_1 \times S_2 \times S_3)/(S_1 \times P_2 \times P_3)$ , where  $V_1$  is the 3-dimensional natural representation of  $S_1 \simeq SL_3$ . In other words it is the vector bundle  $E := \mathcal{O}(1, 1)^{\oplus 3}$  on  $\mathbb{P}^2 \times \mathbb{P}^1$ .

Through 3 points in  $\mathbb{P}^2 \times \mathbb{P}^1$  there passes a rational 2-dimensional family of curves of bidegree  $(2, 1)$ , and the restriction of  $E$  to such a curve is the vector bundle  $\mathcal{O}(3)^{\oplus 3}$ , so that the space of curves passing through 3 points is rational of dimension 5.  $\square$

## 2.4 Some more rationality results

In this subsection, we adress the following natural question raised in [BuMi08]: is the Gromov-Witten variety  $\text{GW}_d(X)$  rational for  $X$  a homogeneous space and  $d$  large enough? As we have seen, in the cominuscule case, this is always true since  $\text{GW}_d(X)$  is even either empty or rational.

We quickly explain how, using the previous techniques, one can prove such results for some rational homogeneous spaces. We only deal with very simple cases where the vector bundles obtained in Proposition 2.1 are well understood.

**Proposition 2.18** *Let  $X$  be an orthogonal Grassmannian, then  $\text{GW}_d(X)$  is rational for large  $d$ .*

*Proof.* We start with orthogonal Grassmannians, so let  $G$  be semi-simple of type  $B_n$  or  $D_n$  and let  $P$  be a maximal parabolic subgroup of  $G$ . By what we already proved, we may assume that  $X = G/P$  is not a quadric and therefore  $X = \mathbb{G}_Q(p, N)$  with  $p > 1$  and  $N = 2n + 1$  or  $2n$ . We can choose the parabolic  $Q$  used in Proposition 2.1 to be maximal and associated to the first simple root (with notation as in [Bou54]). We get an open subset  $Z$  of  $X$  with complement of codimension at least two and a map  $p : Z \rightarrow Y$  where  $Y = G'/P'$  with  $G'$  of the same type as  $G$  with rank one less and  $P'$  maximal. In other words  $Y$  is an orthogonal Grassmannian  $\mathbb{G}_Q(p - 1, N - 2)$ . The vector bundle  $E$  associated to  $p : Z \rightarrow Y$  given by Proposition 2.1 is the tautological quotient bundle on  $Y$ .

Then we know that for a general morphism  $f : \mathbb{P}^1 \rightarrow Y$  of degree at least  $N - p - 1$ , there is no proper vector subspace of the  $N - 1$  dimensional space containing all the subspaces  $f(t)$  for  $t \in \mathbb{P}^1$ . This implies that  $f^*E$  has a decomposition in direct sum of line bundles with positive degrees. In particular if we take a double covering  $g$  of  $f$  we get that  $g^*E$  has a decomposition in direct sum of line bundles with degrees at least 2 and we may conclude by induction on the rank of the group and Proposition 2.3. We therefore have the sufficient bound  $d \geq 2(N - p - 1)$  for the rationality of  $\text{GW}_d(X)$ .  $\square$

Recall that a rational homogeneous space with automorphism group  $G$  is called *adjoint* if it is isomorphic to the closed orbit of the (adjoint) action of  $G$  on the projectivisation of its Lie algebra. The adjoint variety for groups of type  $A_n$  is isomorphic to the point-hyperplane incidence and has therefore Picard group of rank two. We will exclude this case. We also exclude the case of groups of type  $G_2$  for which the method will not work.

**Proposition 2.19** *Let  $X$  be an adjoint variety for a group  $G$  of type different from  $A$  and  $G_2$ , then  $\mathrm{GW}_d(X)$  is rational for large  $d$ .*

*Proof.* We prove this by case by case analysis. Let us remark that for classical groups, the result follows from the previous proposition and the rationality results for cominuscule varieties. We therefore only need to deal with the exceptional cases.

We first remark, by a simple weight computation, that for all adjoint varieties  $X$  such that in the Dynkin diagram of  $\mathrm{Aut}(X)$  there is a cominuscule vertex (corresponding to a cominuscule root), if we look at the open subset  $Z$  of  $X$  and the vector bundle  $p : Z \rightarrow Y$  given by Proposition 2.1, then the associated locally free sheaf  $E$  on  $Y$  is a (non trivial) extension

$$0 \rightarrow T_Y^\vee(1) \rightarrow E \rightarrow \mathcal{O}_Y(1) \rightarrow 0$$

where  $T_Y$  is the tangent bundle of  $Y$ . To prove the result we only need to prove that there exist morphisms  $f : \mathbb{P}^1 \rightarrow Y$  of large degree such that  $f^*(T_Y^\vee(1))$  has a decomposition as a direct sum of line bundles of positive degrees. For this we only deal with types  $E_6$  and  $E_7$  as there is no cominuscule vertex in the Dynkin diagram of the other exceptional groups.

In the first case, the variety  $Y$  is a 10-dimensional spinor variety. Its tangent bundle is  $\Lambda^2 K^\vee$  where  $K$  is the tautological subbundle. If we take a general degree 5 morphism  $f : \mathbb{P}^1 \rightarrow Y$ , then  $f^*(K^\vee) = \mathcal{O}_{\mathbb{P}^1}(2)^5$  thus  $f^*T_Y = \mathcal{O}_{\mathbb{P}^1}(4)^{10}$  and  $f^*(T_Y^\vee(1)) = \mathcal{O}_{\mathbb{P}^1}(1)^{10}$ . Therefore we may apply Proposition 2.3 as soon as  $d \geq 10$  (by taking a double covering of  $f$  as in the proof of the previous proposition).

In the second case, the variety  $Y$  is isomorphic to  $E_6/P_1$ . If we take a general morphism  $f : \mathbb{P}^1 \rightarrow Y$  of degree 4, then  $f^*T_Y = \mathcal{O}_{\mathbb{P}^1}(2)^{16}$  (in fact, from the equality  $[\mathrm{pt}]^3 = q^4$  in the quantum cohomology, see [ChMaPe08], it follows that there is a unique morphism of degree 4 passing through 4 general points, which implies the claim). We get  $f^*(T_Y^\vee(1)) = \mathcal{O}_{\mathbb{P}^1}(1)^{16}$ . Therefore we may apply Proposition 2.3 as soon as  $d \geq 8$ .

For type  $E_8$  groups,  $E_8/P_8$  has dimension 57 and anticanonical class 29. We choose for  $Q$  the maximal parabolic subgroup associated to the simple root  $\alpha_1$ . This root is not cominuscule therefore the construction explained before Proposition 2.1 gives a tower of two morphisms  $Z \xrightarrow{p} W \xrightarrow{q} Y$ , where  $Y$  is isomorphic to a 12-dimensional quadric,  $W$  is the total space of a vector bundle  $E$  over  $Y$ , and  $Z$  is an affine bundle over  $W$  whose associated direction vector bundle is of the form  $q^*F$  with  $F$  a vector bundle on  $Y$  (see [Per02, Proposition 5]). These vector bundles are given by the ascendent central filtration of the unipotent radical of  $Q$ . An easy check on weights gives us two vector bundles  $E$  and  $F$  where  $F$  is as above (therefore of rank 13 and is the tautological quotient bundle on the quadric, of first Chern class 1) and  $E$  is a spinor vector bundle (of rank 32 and first Chern class 16).

We can conclude if we know that for a given integer  $d$ , and for a generic degree  $d$  morphism  $f : \mathbb{P}^1 \rightarrow Y$ , we have  $H^1(\mathbb{P}^1, f^*E) = H^1(\mathbb{P}^1, f^*F) = 0$ . For the vector bundle  $F$ , we argue as in the proof of Proposition 2.18. For  $E$ , this was proved in Lemma 2.11.

Finally, in the case of  $F_4$ , we choose the maximal parabolic subgroup associated to  $\alpha_4$ . The argument is very similar to the  $E_8$ -case: a big open subset of  $F_4/P_1$  (of dimension 15 and anticanonical class 8) is the total space  $Z$  of a tower of affine bundles  $Z \xrightarrow{p} W \xrightarrow{q} Y$ . Here  $Y$  is a

5-dimensional quadric, and, with the same notations as above,  $F$  is the tautological quotient bundle on the quadric  $Y$  (of rank 6 and first Chern class 1) and  $E$  is the spinor bundle on  $Y$  (of rank 4 and first Chern class 2). Then we conclude as for the type  $E_8$ -case.  $\square$

### 3 Quantum to classical principle for equivariant $K$ -theory

#### 3.1 Conditional results of A. Buch and L. Mihalcea

In this section, we will explain a quantum to classical principle for the equivariant quantum  $K$ -theory of a cominuscule homogeneous space  $X$ . For this, we will follow the ideas of A. Buch and L. Mihalcea [BuMi08] who proved such results for the Grassmannian varieties and any degree  $d$ , and for cominuscule varieties and  $d \leq d_{\max}$ , using a construction in [ChMaPe08]. Their technique is valid for any equivariant quantum  $K$ -theoretic invariant (*i.e.*  $d \geq d_{\max}$ ) as soon as the following conjecture is true:

**Conjecture 3.1 (Buch-Mihalcea)** Let  $X$  be a cominuscule variety and let  $d > d_{\max}$ . If  $x_1, x_2, x_3$  are general points in  $X$ , then the Gromov-Witten variety  $GW_d(x_1, x_2, x_3)$  is rational.

We have proved this conjecture in all cases except for one cominuscule homogeneous space (namely  $X = E_6/P_1$ ) and one degree (namely  $d = 3$ ) for which the variety  $GW_d(x_1, x_2, x_3)$  is not rational but empty (see Proposition 2.13). In particular, we will deduce from the general presentation in [BuMi08] a quantum to classical principle for all equivariant quantum  $K$ -theoretic invariants in all these cases except maybe for  $E_6/P_1$  and  $d = 3$ . In Subsection 2.3.2, we also managed to apply their techniques in that case and therefore obtain a general quantum to classical principle.

Before stating our result, let us recall the general picture where the techniques of A. Buch and L. Mihalcea [BuMi08] may apply. We shall assume the following statement about rational curves on the homogeneous space  $X$ :

**Statement 3.2** Let  $d$  be a non negative integer, there exists an element  $w_d$  in  $W^P$  such that

- for a general degree  $d$  morphism  $f : \mathbb{P}^1 \rightarrow X$ , there exists a unique translate of the Schubert variety  $X(w_d)$  containing  $f$ ;
- for three general points in  $X(w_d)$  the variety of degree  $d$  morphisms  $f : \mathbb{P}^1 \rightarrow X(w_d)$  passing through these points is rational.

Once we know that Statement 3.2 is true we can, after A. Buch and L. Mihalcea, make the following construction. First remark that the stabiliser of the Schubert variety  $X(w_d)$  is a parabolic subgroup, say  $Q_d$ , and therefore the variety parametrising the translates of the Schubert variety  $X(w_d)$  is the homogeneous space  $Y_d = G/Q_d$ . We shall denote by  $Z_d$  the incidence variety between  $X$  and  $Y_d$ . We have the following diagram

$$\begin{array}{ccc} Z_d & \xrightarrow{p} & X \\ q \downarrow & & \\ Y_d & & \end{array}$$



and for  $\mu \in Y_d$ , the Schubert variety  $S_\mu$  obtained as the translate of  $X(w_d)$  by  $\mu$  is the fiber  $q^{-1}(\mu)$ . We denote by  $Z_d^{(3)}$  the triple fiber product  $Z_d \times_{Y_d} Z_d \times_{Y_d} Z_d$ , we denote by  $M_d$  the moduli space of degree  $d$  stable maps with three marked points from a rational curve to  $X$  and we define  $\text{Bl}_d$  by

$$\text{Bl}_d = \{(f, S_\mu) \mid f \in M_d, \mu \in Y_d / f \text{ is contained in } S_\mu\}.$$

As in [BuMi08], we have the following diagram:

$$\begin{array}{ccccc} \text{Bl}_d & \xrightarrow{\pi} & M_d & & \\ \phi \downarrow & & \downarrow \text{ev}_i & & \\ Z_d^{(3)} & \xrightarrow{e_i} & Z_d & \xrightarrow{p} & X \\ & & q \downarrow & & \\ & & Y_d & & \end{array}$$

where the map  $\pi$  is just the projection on the first factor, the map  $\phi$  forgets  $f$  but not the three marked points which become points on  $S_\mu$ , the map  $\text{ev}_i$  for  $i = 1, 2$  or  $3$  are the three evaluation maps and  $e_i$  for  $i = 1, 2$  or  $3$  is the map forgetting the two points in  $S_\mu$  of index different from  $i$ .

**Theorem 3.3** *Assume that Statement 3.2 holds, then we have the following formula:*

$$\chi_{M_d}(\text{ev}_1^* \alpha \cdot \text{ev}_2^* \beta \cdot \text{ev}_3^* \gamma) = \chi_{Y_d}(q_* p^* \alpha \cdot q_* p^* \beta \cdot q_* p^* \gamma)$$

where  $\chi_{M_d} : K_T(M_d) \rightarrow K_T(\text{pt})$  is the  $T$ -equivariant Euler characteristic and where  $\alpha, \beta$  and  $\gamma$  are three classes in  $K_T(X)$ .

*Proof.* Remark that if Statement 3.2 holds, then  $\pi$  is birational, because of the uniqueness of the translate of  $X(w_d)$  containing a general curve. The variety  $Z_d^{(3)}$  is a locally trivial fibration above  $Y_d$  with fibres  $X(w_d)^3$ ; since Schubert varieties have rational singularities,  $Z_d^{(3)}$  therefore also has rational singularities. Finally the second point of Statement 3.2 says that a general fibre of  $\phi$  is rational. Thus we may simply reproduce the proof of Theorem 4.2 in [BuMi08].  $\square$

### 3.2 Unconditional results

Let us now explain how to apply the general framework of the previous subsection. We have:

**Proposition 3.4** *Statement 3.2 is true for  $X$  a cominuscule homogeneous space and any degree  $d$ .*

*Proof.* The first point of Statement 3.2 was proved for any cominuscule variety when  $d \leq d_{\max}$  in [BuKrTa03, ChMaPe08] and one can find in *loc.cit.* a description of the integers  $d_{\max}$ , the Schubert varieties  $X(w_d)$ , and the varieties  $Y_d$ . The remaining results have been proved in this article. In fact, let us denote by  $D_{\max}$  the minimal degree for a rational curve to pass through three general points of  $X$ . We can compute  $d_{\max}$  and  $D_{\max}$  as follows:

Type	$X$	$d_{\max}$	$D_{\max}$
$A_{n-1}$	$\mathbb{G}(k, n)$	$\min(p, n - p)$	$\max(p, n - p)$
$B_n, D_n$	$\mathbb{Q}^m$	2	2
$C_n$	$\mathbb{G}_\omega(n, 2n)$	$n + 1$	$n + 1$
$D_{2n}$	$\mathbb{G}_Q(2n, 4n)$	$n$	$n$
$D_{2n+1}$	$\mathbb{G}_Q(2n + 1, 4n + 2)$	$n$	$n + 1$
$E_6$	$E_6/P_1$	2	4
$E_7$	$E_7/P_7$	3	3

For  $d \geq D_{\max}$ , the varieties  $X(w_d)$  and  $Y_d$  are easy to compute:  $X(w_d)$  is simply  $X$  while  $Y_d$  is a point. In particular the first point of Statement 3.2 is trivially true. The only left cases are described in the following array (the results come from [BuMi08] for the Grassmannian variety and Lemmas 2.15 and 2.17 for the  $E_6/P_1$ ) where we assume that  $d \in [d_{\max} + 1, D_{\max} - 1]$ :

<i>Type</i>	$X$	$d$	$X(w_d)$	$Y_d$
$A_{n-1}$	$\mathbb{G}(k, n)$	$d_{\max} < d < D_{\max}$	$\mathbb{G}(d_{\max}, d)$	$\mathbb{G}(d + d_{\max}, n)$
$E_6$	$E_6/P_1$	$d = 3$	$X(s_6 s_5 s_4 s_3 s_1 s_2)$	$E_6/P_4$

The first point of Statement 3.2 is proved in [BuKrTa03] in the case of Grassmannians and is contained in Lemma 2.16 in the case of  $E_6/P_1$ .

For the second point of Statement 3.2, namely the rationality of the Gromov-Witten varieties, we observe that when  $d \leq d_{\max}$  this variety is one point, by [BuKrTa03, ChMaPe08], so assume  $d > d_{\max}$ . The case of Grassmannians of type A was done in [BuMi08]. The case of spinor varieties is done by Corollary 2.6 and Proposition 2.9. The case of symplectic Grassmannians is covered by Theorem 1.6. For  $E_6/P_1$ , we use Lemma 2.17 for  $d = 3$  and Lemma 2.11 for  $d \geq 4$ .  $\square$

By Theorem 3.3, we thus get:

**Theorem 3.5** *For  $X$  a cominuscule homogeneous space and  $d$  any non negative integer, we have the following formula:*

$$\chi_{M_d}(\text{ev}_1^* \alpha \cdot \text{ev}_2^* \beta \cdot \text{ev}_3^* \gamma) = \chi_{Y_d}(q_* p^* \alpha \cdot q_* p^* \beta \cdot q_* p^* \gamma)$$

where  $\chi_{M_d} : K_T(M_d) \rightarrow K_T(\text{pt})$  is the  $T$ -equivariant Euler characteristic and where  $\alpha, \beta$  and  $\gamma$  are three classes in  $K_T(X)$ .

As by definition, the equivariant quantum  $K$ -theoretic Gromov-Witten invariant  $I_d^T(\alpha, \beta, \gamma)$  is  $\chi_{M_d}(\text{ev}_1^* \alpha \cdot \text{ev}_2^* \beta \cdot \text{ev}_3^* \gamma)$ , we deduce the equivariant  $K$ -quantum to classical principle:

**Corollary 3.6** *Let  $X$  be a cominuscule homogeneous space and  $d$  any integer. We have the formula  $I_d^T(\alpha, \beta, \gamma) = \chi_{Y_d}(q_* p^* \alpha \cdot q_* p^* \beta \cdot q_* p^* \gamma)$ .*

For  $d \geq D_{\max}$ , since  $Y_d$  is a point,  $Z_d^{(3)}$  is equal to  $X \times X \times X$ , and we get (compare with Consequence 7.2 in [BuMi08]):

**Corollary 3.7** *For  $d \geq D_{\max}$ , we have the formula  $I_d^T(\alpha, \beta, \gamma) = \chi_X(\alpha) \cdot \chi_X(\beta) \cdot \chi_X(\gamma)$ .*

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